Approximate Wiener-Hopf factorization for Finance Problems[1]

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ABSTRACT

The Wiener-Hopf Factorization (WHF) is a powerful mathematical tool in the study of various functionals depending on a Lévy process. In Lévy processes arising from financial problems, typical quantities of interest are the extrema, first/last passage time, the overshoot, and the last time the extrema were archived. Similar questions arise when pricing options, etc. Unfortunately, in many situations the WHF problem cannot be solved explicitly. This article studies the problem of approximating, in $L^2(\mathbb{R})$, a WHF. We provide a procedure to solve a WHF problem approximately, and provide error bounds for the approximation process. As an application, we demonstrate two procedures for finding the distributions of extrema of a Lévy process.

Keywords: Principal value integral; Padé approximant; continued fraction; Fourier transform; Hilbert transform, mathematical finance

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1. Introduction

The Wiener-Hopf Factorization (WHF) is a powerful mathematical tool in study of various functionals of a Lévy process, such as extrema, first/last passage time and the overshoot, the last time the extrema was archived, etc. These results are very important from the theoretical point of view, for example they can be used to prove a general theorem about short/long time behavior of a process, see Bertoin (1996) and Kyprianou (2006). In recent years, there is also a growing interest in applications of WHF in mathematical finance, where the above mentioned functionals are being used to describe the payoff of a contract and the corresponding probability distribution is used to compute its price, see Asmussen, et al. (2004), Boyarchenko & Levendorskiĭ (1999, 2002a, 2002b, 2002c, 2002d), Mordecki (2002), Schoutens (2006), Cardi (2005), and Green, et al. (2010), among others.

Roughly speaking, the homogeneous case of the Wiener- Hopf factorization problem is the problem of finding an analytic and bounded complex-valued function Φ that has a prescribed jump discontinuity across the real line $\mathbb{R} \subset \mathbb{C}$. It is usual to denote the radial limit from above at the real line by Φ^+ and the corresponding radial limit from below by Φ^-. Thus, Φ^+ and Φ^- are functions on the real line such that $\Phi^+(\omega)\Phi^-(\omega) = g(\omega)$, where g is a given function with suitable properties, see below. There exists a general solution in terms of Sokhotskyi-Plemelj integrals, as will be discussed later, but explicit elementary formulas for $\Phi^\pm$ can rarely be found.

There are several approximation techniques in literature, see Kudryavtsev & Levendorskiĭ (2009), Green, et al. (2010), and Kuzentsov (2009) among others. But, as far as we known, (**i**) none of them provide estimation bound for their approximation methods; (**ii**) some of them need uniform convergence of the approximations within a strip around the real line which is usually hard to achieve.

This article studies the problem of solving a WHF problem, approximately. Then, (**i**) we provide an $L^2(\mathbb{R})$, approximation for a WHF problem; (**ii**) we provide error bounds for approximation; (**iii**) we employ the technique to approximate the distribution of extrema of a Lévy process; and (**iv**) we introduce two procedures to generate appropriate approximate solutions for the distribution of extrema of a Lévy process.

Section 2 contains lemmas and propositions required for the other sections. It provides an $L^2(\mathbb{R})$, approximation technique for solving a WHF problem. Section 3 considers the problem of finding the density functions for the extrema of a Lévy process. Then, it develops two approximate techniques for situations where those density functions cannot be determined, explicitly. Error bounds on the approximation techniques are given. Several examples have been solved, in Section 4, to provide applications of the results.

2. Preliminaries

The *Sokhotskyi-Plemelj integral* of a function s which satisfies the Hölder condition and it is defined by a principal value integral, as follows.

$$\phi_s(\lambda) := \frac{1}{2\pi i} \fint_{\mathbb{R}} \frac{s(x)}{x-\lambda} dx, \quad \text{for } \lambda \in \mathbb{C}. \tag{1}$$

The following lemma provides the definition and some well known properties of the radial limits of a Sokhotskyi-Plemelj integral, see Gakhov (1990) and Pandey (1996), among others, for more detail.

Lemma 1. *The radial limits of the Sokhotskyi-Plemelj integral of s, are given by $\phi_s^{\pm}(\omega) = \lim_{\lambda \to \omega + i0^{\pm}} \phi_s(\lambda)$ and satisfy a jump formula:*

i) $\phi_s^{\pm}(\omega) = \pm s(\omega)/2 + \phi_s(\omega)$, *for* $\omega \in \mathbb{R}$;

ii) $\phi_s^{\pm}(\omega) = \pm s(\omega)/2 + H_s(\omega)/(2i)$, *where* $H_s(\omega)$ *stands for the* Hilbert transform *of* s *and* $\omega \in \mathbb{R}$.

The WHF is the problem of finding a function Φ, that is analytic and bounded except on the real line, where its upper and lower radial limits $\Phi^{\pm}$ satisfy

$$\Phi^{+}(\omega)\Phi^{-}(\omega) \quad = \quad g(\omega),\ \omega \in \mathbb{R}. \tag{2}$$

The function g is given. We suppose g is continuous, positive, satisfies a Hölder condition on $\mathbb{R}$, has zero index, has $g(0) = 1$, is nowhere zero on $\mathbb{R}$, and bounded by 1.

The following provides solutions for the above WHF problem. We begin with what we term the Resolvent Equation for Sokhotskyi-Plemelj integrals.

Lemma 2. *The Sokhotskyi-Plemelj integral of a function f satisfies*

$$\phi_f(\lambda) - \phi_f(\mu) = (\lambda - \mu)\phi_{\frac{f(x)}{x-\lambda}}(\mu)$$

for λ and μ real or complex.

Proof. In general,

$$(x-\lambda)^{-1} - (x-\mu)^{-1} = (\lambda-\mu)(x-\mu)^{-1}(x-\lambda)^{-1}.$$

Then, see Dunford & Schwartz (1988), we have an equation of Cauchy integrals, where $\Gamma = \mathbb{R}$:

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(x)}{x-\lambda}dx - \frac{1}{2\pi i}\int_\Gamma \frac{f(x)}{x-\mu}dx = \frac{\lambda-\mu}{2\pi i}\int_\Gamma \frac{f(x)}{(x-\mu)(x-\lambda)}dx.$$

□

The above is valid only for λ and μ not on the real line. However, by Lemma 1 the values of ϕ_f on the real line are obtained by averaging the limit from above, ϕ_f^+, and the limit from below, ϕ_f^-. We thus obtain the stated equation in all cases.

Lemma 3. *Suppose $\Phi^{\pm}$ are sectionally analytic functions that satisfy the WHF problem given by 2. Moveover, suppose that g is a zero index function which satisfies the Hölder condition and $g(0) = 1$.*[3] *Then*

$$\Phi^{\pm}(\lambda) \quad = \quad \exp \pm\{\phi_{\ln g}(\lambda) - \phi_{\ln g}(0)\},$$

[3]The condition $g(0) = 1$ does not always hold in financial applications of the WHF, but happen to arise in our application, and can lead to complications. Lemma 3 is used to simplify in this case.

where $\phi_{\ln g}$ stands for the Sokhotskyi-Plemelj integral of $\ln g$.

Proof. Using a suggestion in Gakhov (1990) for solving the homogeneous Riemann-Hilbert problem in equation (2) gives, see also Lewis & Mordecki(2008):

$$\Phi^{\pm}(\lambda) \quad = \quad \exp\{\pm\frac{\lambda}{2\pi i}\int_{\mathbb{R}}\frac{\ln g(x)/x}{x-\lambda}dx\}.$$

Lemma 2 with $f = \ln g$ gives

$$\phi_{\ln g}(\lambda) - \phi_{\ln g}(\mu) = (\lambda-\mu)\phi_{\frac{\ln g(x)}{x-\lambda}}(\mu).$$

Letting λ go to zero from above, in the complex plane, and using the fact that $\ln g(0) = 0$, Lemma 1 lets us conclude that

$$\phi_{\ln g}(0) - \phi_{\ln g}(\mu) = -\mu\phi_{\frac{\ln g(x)}{x}}(\mu).$$

Substituting this into the above equation for $\Phi^{\pm}$ gives our claimed result. $\square$

The following explores some properties of the above lemma.

Remark 1. *Using the jump formula one can conclude that*

$$\Phi^{\pm}(\omega) \quad = \quad \sqrt{g(\omega)}\exp\{\pm\frac{i}{2}(H_{\ln g}(0) - H_{\ln g}(\omega))\},$$

where $H_{\ln g}$ stands for the Hilbert transform of $\ln g$.

There is a situation which one may evaluate solutions of the WHF problem 2 directly, rather than using the Sokhotskyi-Plemelj integrations.

Remark 2. *If g can be decomposed as a product of two sectionally analytic functions g^+ and g^-, respectively in $\mathbb{C}^+ = \{\lambda : \ \lambda \in \mathbb{C}$ and $\Im(\lambda) \geq 0\}$ and $\mathbb{C}^- = \{\lambda : \ \lambda \in \mathbb{C}$ and $\Im(\lambda) \leq 0\}$. Then, solutions of the WHF problem 2 are given by $\Phi^+ \equiv g^+$ and $\Phi^- \equiv g^-$.*

The above technique is well known as Carlemann's method, but it is really just a restatement of the Riemann-Hilbert problem, and thus is only useful if there is a way to find the required functions.

Remark 3. *If g is a rational function $\frac{P(x)}{Q(x)}$ that has no poles or zeros on the real line, then the Riemann-Hilbert factorization problem can be solved. We factor the polynomials P and Q, and then let P^+ be the product of those factors of P that have zeros in the lower half plane, and P^- be the product of those factors that have zeros in the upper half plane. Then, setting $g^+ = \frac{P^+(x)}{Q^+(x)}$ and $g^- = \frac{P^-(x)}{Q^-(x)}$ gives us (up to a scalar multiple) our desired factorization.*

An useful result for the two next sections is the *Hausdorff-Young* theorem, a proof may be found in Pandey (1996).

Theorem 1. *(Hausdorff-Young) If s is a function in $L^p(\mathbb{R})$, then the $L^p(\mathbb{R})$ and $L^{p^*}(\mathbb{R})$-norms of s and the Fourier transform of s, say $\hat{s}$, respectively satisfy $||\hat{s}||_{p^*} \leq (2\pi)^{-1/p}||s||_p$, where $1 \leq p \leq 2$ and $1/p + 1/p^* = 1$.*

From the Hausdorff-Young Theorem, one can observe that if $\{s_n\}$ is a sequence of functions converging in $L^p(\mathbb{R})$, $1 \leq p \leq 2$, to s., then the Fourier transforms of the s_n converges in $L^{p^*}(\mathbb{R})$, to the Fourier transform of s, where $1/p + 1/p^* = 1$. The converse is false.

A similar property of the Hilbert transform is.

Lemma 4. *(Titmarsh-Riesz Theorem) Suppose s is a function in $L^p(\mathbb{R})$, $1 \leq p \leq 2$. Then, $||H_s||_p \leq ||s||_p$, where H_s stands for the Hilbert transform of s.*

Using the above lemma, one may conclude that if $\{f_n\}$, is a sequence of functions which converge, in $L^2(\mathbb{R})$, to f. Then, the Hilbert transforms $H(f_n)$ converge, in $L^2(\mathbb{R})$, to the Hilbert transform of f.

The following, from Kucerovsky & Payandeh (2009), recalls some further useful properties of functions in $L^2(\mathbb{R})$ space.

Lemma 5. *Suppose s and r are functions in $L^2(\mathbb{R})$. Then,*

i) $||\sqrt{s} - \sqrt{r}||_2 \leq \frac{1}{2\sqrt{a}}||s - r||_2$, *whenever $|s|$ and $|r|$ are bounded above by a;*

ii) $||\ln s - \ln r||_2 \leq a^{-1}||s-r||_2$, *whenever s and r are positive-valued functions and $|s|$ and $|r|$ are bounded above by a;*

iii) $||e^{-is/2} - e^{-ir/2}||_2 \leq \frac{1}{2}||s-r||_2$, *whenever s and r are real-valued functions;*

iv) $||1/s - 1/r||_2 \leq a^{-2}||s-r||_2$, *whenever $|s|$ and $|r|$ are bounded above by a.*

In many situations, the WHF problem (Equation 2) cannot be solved explicitly and has to be solved approximately, see Kucerovsky & Payandeh (2009), for more detail. The following develops an approximate technique to solve a WHF problem of the form in Equation 2.

Theorem 2. *Suppose $\Phi^{\pm}$ are sectionally analytic functions satisfying the WHF problem of equation 2 where*

A_1) *g is real, positive, and bounded above by 1, of index zero, satisfies the Hölder condition and $g(0) = 1$;*

A_2) *the sequence g_n converges, in $L^2(\mathbb{R})$, to g.*

Then, $\Phi^{\pm}$ can be approximated by $\Phi_n^{\pm}$, where

$$||\Phi_n^{\pm} - \Phi^{\pm}||_2 \quad \leq \quad \frac{1}{2}||g_n - g||_2^2 + \frac{3}{2}||g_n - g||_2,$$

and a is an upper bound for g.

Proof. Set $k(\omega) := -H_{\ln g}(\omega) + H_{\ln g}(0)$ and $k_n(\omega) := -H_{\ln g_n}(\omega) + H_{\ln g_n}(0)$. Now, from Remark 1 observe that

$$\begin{aligned}
||\Phi_n^{\pm} - \Phi^{\pm}||_2 &= ||\sqrt{g_n}e^{\pm ik_n/2} - \sqrt{g}e^{\pm ik/2}||_2 \\
&\leq [||\sqrt{g_n} - \sqrt{g}||_2 + ||\sqrt{g}||_2]\, ||e^{\pm ik_n/2} - e^{ik/2}||_2 + |e^{\pm ik/2}|||\sqrt{g_n} - \sqrt{g}||_2 \\
&\leq \frac{1}{2}[||\sqrt{g_n} - \sqrt{g}||_2 + ||\sqrt{g}||_2]\, || - H_{\ln g_n}(\omega) + H_{\ln g_n}(0) + H_{\ln g}(\omega) - H_{\ln g}(0)||_2 \\
&\quad + |e^{\pm ik/2}|||\sqrt{g_n} - \sqrt{g}||_2 \\
&\leq [||\sqrt{g_n} - \sqrt{g}||_2 + ||\sqrt{g}||_2]\, ||H_{\ln g_n} - H_{\ln g}||_2 + ||\sqrt{g_n} - \sqrt{g}||_2 \\
&\quad \text{since } k \text{ and } k_n \text{ are real-valued functions} \\
&\leq [||\sqrt{g_n} - \sqrt{g}||_2 + ||\sqrt{g}||_2]\, ||\ln(g_n) - \ln(g)||_2 + ||\sqrt{g_n} - \sqrt{g}||_2 \\
&\leq \left[\frac{1}{2}||g_n - g||_2 + 1\right] ||g_n - g||_2 + \frac{1}{2}||g_n - g||_2 \\
&= \frac{1}{2}||g_n - g||_2^2 + \frac{3}{2}||g_n - g||_2. \ \square
\end{aligned}$$

Remark 4. *By Remark 3 approximation by rational functions is desirable because it leads to relatively easy to solve approximate problems. Approximation of g a sequence of rational functions can be done by Padé approximation or preferably by a continued fraction expansion.*

We now recall the definition and some useful properties of a mixed gamma distribution, which plays an important role for the next sections, see Bracewell (2000) for more detail.

Definition 1. *(mixed gamma family of distributions) A nonnegative random variable X is said to be distributed according to a mixed gamma distribution if its density function is given by*

$$p(x) = \sum_{k=1}^{\nu}\sum_{j=1}^{n_\nu} c_{kj}\frac{\alpha_k^j x^{j-1}}{(j-1)!}e^{-\alpha_k x}, \ x \geq 0, \tag{3}$$

where c_{k_j} and α_k are positive value which satisfy $\sum_{k=1}^{\nu}\sum_{j=1}^{n_\nu} c_{k_j} = 1$.

Lemma 6. *The characteristic function $\hat{p}$ of a distribution (mathematically it is the Fourier transform of its density function) has the following properties:*

i) *$\hat{p}$ is a rational function if and only if the density function belongs to the mixed gamma family given by 3;*

ii) $\hat{p}(0) = 1$; *and the norm of* $\hat{p}(\omega)$ *is bounded by 1.*

The next section provides an application of Theorem 2 to the problem of finding the distributions of the extrema of Lévy process X_t, approximately.

3. Application to extrema's distributions of a Lévy process

Suppose that X_t is a one-dimensional real-valued Lévy process starting from $X_0 = 0$ and defined by a triplet (μ, σ, ν), where $\mu \in \mathbb{R}$ is the drift, $\sigma \geq 0$ is the volatility, and the jumps measure is ν, given by a nonnegative function defined on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}} \min\{1, x^2\}\nu(dx) < \infty$. Moreover, suppose that the stopping time $\tau(q)$ has either a geometric or an exponential distribution with parameter q independent of the Lévy process X_t and $\tau(0) = \infty$. The Lévy-Khintchine representation ensures that the characteristic exponent ψ (i.e., $\psi(\omega) = \ln(E(\exp(i\omega X_1))),\ \omega \in \mathbb{R}$) can be represented by

$$\psi(\omega) \;=\; i\mu\omega - \frac{1}{2}\sigma^2\omega^2 + \int_{\mathbb{R}} (e^{i\omega x} - 1 - i\omega x I_{[-1,1]}(x))\nu(dx), \quad \omega \in \mathbb{R}. \tag{4}$$

The extrema of the Lévy process X_t are given by

$$\begin{aligned} M_q &= \sup\{X_s : \ s \leq \tau(q)\} \\ I_q &= \inf\{X_s : \ s \leq \tau(q)\}. \end{aligned} \tag{5}$$

The WHF is a well known technique for studying the characteristic functions of the extrema random variables (Bertoin; 1996). Namely, the WHF states that:

(i) the random variables M_q and I_q are independent;

(ii) the product of their characteristic functions is equal to the characteristic function of the Lévy process X_t;

(iii) the random variable M_q (I_q) is infinitely divisible, positive (negative), and has zero drift.

In the case that the characteristic function of a Lévy process X_t can be decomposed as a product of two sectionally analytic functions in the closed upper and lower half complex planes $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively, the characteristic functions of random variables M_q and I_q can be determined explicitly. This occurs, for example, if the characteristic function is a rational function. Lewis & Mordecki (2005) considered a Lévy process X_t whose negative jumps were distributed according to a mixed gamma family of distributions and an arbitrary positive jumps measure. They established such a process has a characteristic function that can be decomposed as a product of a rational function and a more general function, which however is analytic in $\mathbb{C}^-$, respectively. Recently, they provided an analogous result for a Lévy process whose corresponding positive jumps measure comes from a mixed gamma family of distributions while its negative jumps measure is fairly arbitrary, more detail can be found in Lewis & Mordecki (2008).

Unfortunately, in the majority of situations, the characteristic function of the process neither is a rational function nor can be decomposed in any evident way as a product of two analytic functions in $\mathbb{C}^+$ and $\mathbb{C}^-$. An alternative is to obtain the characteristic functions of M_q and I_q in terms of a Sokhotskyi-Plemelj singular integral (see Equation 1). But, this form, also, presents some difficulties in numerical work due to slow evaluation and numerical problems caused by singularities near the integral contour. To overcome these difficulties, approximation methods have to be considered.

It is well known that a Lévy process X_t whose jumps distribution arises from a phase-type distribution has a rational characteristic function (Doney; 1987). Kuznetsov (2009) utilized this fact and approximated a jumps measure ν by a ten-parameter family of Lévy processes (a so-called $\beta-$family of Lévy processes) using a sequence of phase-type measures. Then, he obtained approximations for the characteristic functions of random variables M_q and I_q, respectively.

The following theorem represents an error bound for approximating density functions of extrema of a Lévy process.

Theorem 3. *Suppose X_t is a Lévy process defined by a triple (μ, σ, ν). Moreover, suppose that*

A_1) *the stopping time $\tau(q)$ is either a geometric or an exponential distribution with parameter q independent of X_t and $\tau(0) = \infty$;*

A_2) *the $\nu_n(dx)$ are a sequence of density functions which converge in $L^2(\mathbb{R})$, to ν and $\int_{-1}^{1} x\nu_n(dx) = \int_{-1}^{1} x\nu(dx)$.*

Then, the distribution of the suprema and infima the of Lévy process X_t, denoted f_q^+ and f_q^-, respectively, can be approximated by a sequence of density functions $f_{q,n}^+$ and $f_{q,n}^-$ where:

i) *For exponentially distributed stopping time $\tau(q)$,*

$$||f_q^{\pm} - f_{q,n}^{\pm}||_2 \le \frac{1}{q^2\sqrt{8\pi}}||\nu_n - \nu||_2^2 + \frac{3}{2q}||\nu_n - \nu||_2;$$

ii) *For geometric stopping time $\tau(q)$,*

$$||f_q^{\pm} - f_{q,n}^{\pm}||_2 \le \frac{(1-q)^2}{q^2\sqrt{8\pi}}||\nu_n - \nu||_2^2 + \frac{3(1-q)}{2q}||\nu_n - \nu||_2.$$

Proof. From Bertoin (1996), one can observe that the Fourier transform of M_q and I_q density functions, say respectively Φ^+ and Φ^-, satisfy *either* the WHF $\Phi^+(\omega)\Phi^-(\omega) = q/(q - \psi(\omega)$, where $\omega \in \mathbb{R}$ (for exponentially distributed stopping time) *or* the WHF $\Phi^+(\omega)\Phi^-(\omega) = (1-q)/(1 - q\exp\{-\psi(\omega)\}$, where $\omega \in \mathbb{R}$ (for geometric stopping time). Now, from the fact that the expressions $q(q - \psi(\cdot))^{-1}$ and $(1-q)(1 - q\exp\{-\psi(\cdot)\})^{-1}$ are the characteristic function of the Lévy process X_t, for exponential and geometric stopping time, respectively, we observe that both expressions are bounded by 1 because of the property of the characteristic function given by Lemma 6, part ii). For part (i), from Theorem 2 observe that

$$\begin{aligned}
||\Phi_n^{\pm} - \Phi^{\pm}||_2 &\le \frac{1}{2}||\frac{q}{q-\psi_n} - \frac{q}{q-\psi}||_2^2 + \frac{3}{2}||\frac{q}{q-\psi_n} - \frac{q}{q-\psi}||_2 \\
&\le \frac{1}{2q^2}||\psi_n - \psi||_2^2 + \frac{3}{2q}||\psi_n - \psi||_2 \\
&\le \frac{1}{4\pi q^2}||\nu_n - \nu||_2^2 + \frac{3}{q\sqrt{8\pi}}||\nu_n - \nu||_2.
\end{aligned}$$

The second inequality arrives from the fact that the characteristic function $q(q-\psi(\cdot))^{-1}$ is bounded above by 1, while the third inequality comes from the Levy-Khintchine representation (Equation 4) along with conditions A_2 and the Hausdorff-Young Theorem (Lemma 1). The rest of proof arrives from an application of the Hausdorff-Young Theorem (Lemma 1). The proof of part (ii) is quite similar. $\square$

Remark 5. *In the case of* $\int_{-1}^{1} x\nu_n(dx) =: c_n \neq d := \int_{-1}^{1} x\nu(dx)$*, one may obtain a sequence* $\xi_n = \frac{d}{c_n}\nu_n$ *which satisfies the desired condition.*

The following utilizes Kuznetsov (2009), Lewis & Mordecki (2005), and Lewis & Mordecki (2008) observation to provide a procedure to find the distribution of extrema of a wide class of Lévy processes, approximately.

Procedure 1. *Suppose* X_t *is a Lévy process with bounded above characteristic exponents* ψ*. Moreover, suppose that the stopping time* $\tau(q)$ *is either a geometric or an exponential distribution with parameter* q *independent of* X_t *and* $\tau(0) = \infty$*. Then, by the following steps one can approximate in* $L^2(\mathbb{R})$*, the density functions of the extrema random variables* M_q *and* I_q*.*

Step 1. *Approximate the jumps measure* ν *with either a phase-type or a mixed gamma density function, say* ν^**, where* $\int_{-1}^{1} x\nu^*(dx) = \int_{-1}^{1} x\nu(dx)$ *and* $||\nu - \nu^*||_2 \leq \varepsilon$*;*

Step 2. *decompose the rational function* $q/(q-\psi(\omega))$ *(or* $(1-q)/(1-q\exp\{-\psi(\omega)\})$*) into a product of two rational and sectionally analytic functions, say* $g^{\pm}$*, in* $\mathbb{C}^{\pm}$*, respectively;*

Step 3. *obtain, approximately, density functions of* M_q *and* I_q *by the inverse Fourier transform of* g^+ *and* g^-*, respectively.*

In many situations, it is more continent to approximate the characteristic exponents ψ, rather than the jumps measure ν. The following extends the results of Theorem 3 to such situations.

Corollary 1. *Suppose* X_t *is a Lévy process defined by a triple* (μ, σ, ν)*. Moreover, suppose that*

A_1) *the stopping time $\tau(q)$ is either a geometric or an exponential distribution with parameter q independent of X_t and $\tau(0)=\infty$;*

A_2) *the exponents ψ_n are a sequence of functions which converge in $L^2(\mathbb{R})$, to the characteristic exponent of the process ψ; and $\int_{-1}^{1} x\nu_n(dx) = \int_{-1}^{1} x\nu(dx)$.*

Then, the distribution of the supremum and the infimum of Lévy process X_t, say respectively f_q^+ and f_q^- can be approximated by a sequence of density functions $f_{q,n}^+$ and $f_{q,n}^-$ where:

i) *For exponentially distributed stopping time $\tau(q)$,*

$$||f_q^\pm - f_{q,n}^\pm||_2 \le \frac{\sqrt{2\pi}}{2q^2}||\psi_n-\psi||_2^2 + \frac{3\sqrt{2\pi}}{2q}||\psi_n-\psi||_2;$$

ii) *For discrete stopping time $\tau(q)$,*

$$||f_q^\pm - f_{q,n}^\pm||_2 \le \frac{(1-q)^2\sqrt{2\pi}}{2q^2}||\psi_n-\psi||_2^2 + \frac{3(1-q)\sqrt{2\pi}}{2q}||\psi_n-\psi||_2.$$

Using the fact that the characteristic exponent $\psi(i\omega)$, $\omega\in\mathbb{R}$, is a real-valued function, (Bertoin; 1996), one can suggest the following procedure to generate approximation density functions for M_q and I_q:

Procedure 2. *Suppose X_t is a Lévy process with bounded, above, characteristic exponents ψ. Moreover, suppose that stopping time $\tau(q)$ is either a geometric or an exponential distribution with parameter q independent of X_t and $\tau(0)=\infty$. Then, by the following steps, one can approximate in $L^2(\mathbb{R})$, sense density functions of the extrema random variables M_q and I_q.*

Step 1. *Approximate in $L^2(\mathbb{R})$, the characteristic exponent $\psi(i\omega)$ with a rational function, $\psi^*(i\xi)$; generated by a Padé approximant or the continued fraction for the characteristic exponent,*

Step 2. *decompose the rational function $q/(q-\psi^*(\omega))$ (or $(1-q)/(1-q\exp\{-\psi^*(\omega)\})$) into the product of two rational and sectionally analytic functions, $g^\pm$, in $\mathbb{C}^\pm$, by Remark 3,*

Step 3. *obtain density functions of M_q and I_q by the inverse Fourier transform of g^+ and g^-, respectively.*

Using Lemma 6, one can readily conclude that the above two procedures approximate density functions of M_q and I_q by the mixed gamma density functions.

4. Examples

The following examples explore the applications of the results of the previous sections.

Example 1. *Kuznetsov (2009) considered a compound Poisson process with a jumps measure $\nu(dx) = \exp\{\alpha x\} sech(x)dx$, and showed the characteristic exponent for such a compound Poisson process is given by*

$$\psi(\omega) = \frac{\pi}{\cos(\pi\alpha/2)} - \frac{\pi}{\cosh(\pi(\omega - i\alpha)/2)}, \ \omega \in \mathbb{R}.$$

He established that over the whole complex plane an expression $q(q - \psi(\cdot))^{-1}$ can be uniformly approximated by a product $\rho_q^+(\cdot)\rho_q^-(\cdot)$, where

$$\begin{aligned} \rho_q^+(\lambda) &= \prod_{n=0}^{\infty} \frac{(1 - \frac{i\lambda}{4n+1-\alpha})(1 - \frac{i\lambda}{4n+3-\alpha})}{(1 - \frac{i\lambda}{4n+\eta-\alpha})(1 - \frac{i\lambda}{4n+4-\eta-\alpha})}; \\ \rho_q^-(\lambda) &= \prod_{n=0}^{\infty} \frac{(1 + \frac{i\lambda}{4n+1+\alpha})(1 + \frac{i\lambda}{4n+3+\alpha})}{(1 + \frac{i\lambda}{4n+\eta+\alpha})(1 + \frac{i\lambda}{4n+4-\eta+\alpha})}, \end{aligned}$$

where $\lambda \in \mathbb{C}$ and $\eta = 2/\pi \arccos(\pi/(q + \pi\sec(\alpha\pi/2)))$. Therefore, approximate solutions for $\Phi_q^{\pm}$ are given by $\rho_q^{\pm}$, more detail can be found in Kuznetsov (2009).

A Lévy process X_t with independent and exponentially distributed stopping time $\tau(q)$ is said to be in a $\beta-$family of Lévy processes if its corresponding jumps measure is given by

$$\nu(dx) = c_1 \frac{e^{-\alpha_1\beta_1 x}}{(1 - e^{-\beta_1 x})^{\lambda_1}} I_{(0,\infty)}(x)dx + c_2 \frac{e^{\alpha_2\beta_2 x}}{(1 - e^{-\beta_2 x})^{\lambda_2}} I_{(-\infty,0)}(x)dx, \tag{6}$$

where the parameters satisfy $\alpha_i,\ \beta_i > 0$, $c_i \geq 0$, and $\lambda_i \in (0,3)$. Many well known Lévy processes such as the generalized tempered stable processes (with $c_i = c_i^* \beta^{\lambda_i},\ \alpha_i = \alpha_i^* \beta^{-1},\ \beta_i = \beta \to 0^+$), the KoBol processes (with $\lambda_1 = \lambda_2$), and the CGMY processes (with $c_1 = c_2,\ \lambda_1 = \lambda_2$, and $\beta_1 = \beta_2$) are some well known examples of such Lévy processes, see Kuznetsov (2009) for more detail.

Example 2. *Suppose X_t is a $\beta-$Lévy process with a jumps measure $\nu(dx)$, given by Equation 6. Kuznetsov (2009) established the characteristic exponent for such process, whenever $\lambda_1 \in (0,3) \setminus \{1,2\}$, in terms of the Beta and digamma, Psi, functions, as the following.*

$$\psi(\omega) \quad = \quad i\rho\omega + \frac{\sigma^2\omega^2}{2} - \frac{c_1}{\beta_1} Beta(\alpha_1 - i\omega/\beta_1;\ 1-\lambda_1) - \frac{c_2}{\beta_2} Beta(\alpha_2 - i\omega/\beta_2;\ 1-\lambda_2) + \gamma,$$

where $\omega \in \mathbb{R}$, $\gamma = c_1 Beta(\alpha_1;\ 1-\lambda_1)/\beta_1 + c_2 Beta(\alpha_2;\ 1-\lambda_2)/\beta_2$, $\rho = c_1 Beta(\alpha_1;\ 1-\lambda_1)[Psi(1+\alpha_1-\lambda_1) - Psi(\alpha_1)]/\beta_1^2 + c_2 Beta(\alpha_2;\ 1-\lambda_2)[Psi(1+\alpha_2-\lambda_1) - Psi(\alpha_2)]/\beta_2^2 - \mu$, and μ and σ are given. Kuznetsov (2009). showed that an expression $q(q-\psi(\lambda))^{-1},\ \lambda \in \mathbb{C}$, may be, uniformly, approximated by

$$(1+\frac{i\lambda}{\xi_0^+})^{-1}(1+\frac{i\lambda}{\xi_0^-})^{-1}\prod_{n\geq 1}\frac{1+\frac{i\lambda}{\beta_2(n+\alpha_2)}}{1+\frac{i\lambda}{\xi_n}}\prod_{n\leq -1}\frac{1+\frac{i\lambda}{\beta_1(n-\alpha_1)}}{1+\frac{i\lambda}{\xi_n}},$$

where $\xi_0^\pm$ and ξ_n are roots of $q-\psi(i\xi)=0,\ \xi \in \mathbb{C}$. The approximate solution for $\Phi^\pm$ can be found by decomposing the above expression as a product of two functions that are analytic and bounded in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively.

Example 3. *Consider a 1-stable Lévy process with a jumps measure $\nu(dx) = c|x|^{-2} I_{(-\infty,\infty)}(x)dx$. One can, readily, show that the characteristic exponent of such process is $\psi(\omega) = (c_1+c_2)|\omega|\{1 + i\beta sgn\omega\frac{2}{\pi}\ln(|\omega|)\} + i\omega\eta$, where η is a normalized real-valued. The natural logarithm $\ln(|\lambda|)$ has the continued fraction*

$$\ln(|\lambda|) = \frac{\lambda-1}{1}{}_{+}\frac{1^2(\lambda-1)}{2}{}_{+}\frac{1^2(\lambda-1)}{3}{}_{+}\frac{2^2(\lambda-1)}{4}{}_{+}\frac{2^2(\lambda-1)}{5}{}_{+}\frac{3^2(\lambda-1)}{7}{}_{+}\frac{3^2(\lambda-1)}{9}{}_{+}\cdots,\ \lambda \in \mathbb{C}.$$

(see Jones & Thron; 1931 or Cuyt et al; 2008, among others). By truncating the above continued fraction one obtains a very well-behaved rational approximation for $\ln(\cdot)$. Consequently, an expression $q(q-\psi(\cdot))^{-1}$ can be approximated by rational function $P(\lambda)/Q(\lambda)$. Now, the characteristic

functions for the extrema can be obtained after decomposing $P(\lambda)/Q(\lambda)$ into product of two rational, analytic, and bounded functions, in $\mathbb{C}^+$ and $\mathbb{C}^-$, as in Remark 3.

Example 4. *Suppose X_t is a Lévy process with independent and continuous $\tau(q)$ and a jumps measure $\nu(dx) = \exp\{\alpha x\} cosech^2(x/2)dx$. The characteristic exponent for such Lévy process is given by*

$$\psi(\omega) \quad = \quad \frac{\sigma^2\omega^2}{2} + i\rho\omega + 4\pi(\omega - i\alpha)\coth(\pi(\omega - i\alpha)) - 4\gamma,$$

where $\gamma = \pi\alpha\cot(\pi\alpha)$, $\rho = 4\pi^2\alpha + \frac{4\gamma(\gamma-1)}{\alpha} - \mu$, $\omega \in \mathbb{R}$, and α, μ, and σ are given. The continued fraction for $\tanh(\lambda)$ is given by

$$\tanh(\lambda) \quad = \quad \frac{\lambda}{1} + \frac{\lambda^2}{3_{\,+}}\frac{\lambda^2}{5_{\,+}}\frac{\lambda^2}{7_{\,+}}\frac{\lambda^2}{9_{\,+}}\cdots, \ \lambda \in \mathbb{C},$$

(Cuyt et al; 2008). After cutting off the above continued fraction at the n^{th} term, one obtains a rational function, say ρ_n, for $\tanh(\cdot)$. Substituting the ρ_n in an expression $q(q - \psi(\cdot))$, we obtain a rational function, say $P(\lambda)/Q(\lambda)$. Therefore, the characteristic functions for the extrema can be found by decomposing a rational function $P(\lambda)/Q(\lambda)$ as a product of two rational, analytic, and bounded functions in $\mathbb{C}^+$ and $\mathbb{C}^-$. This observation verifies Kuznetsov (2009)'s result.

Example 5. *The Metron model is a Lévy process with a jumps measure $\nu(x) = a(\delta\sqrt{2\pi})^{-1}\exp\{-(x-\mu)^2/(2\delta^2)\}$ and characteristic exponent $\psi(\omega) = i\mu\omega - \sigma^2\omega^2/2 + a\{e^{-\delta^2\omega^2/2+i\mu\omega} - 1\}$, where a, δ, μ, and σ are given and $\omega \in \mathbb{R}$. Metron model has tail behavior that is heavier than the Gaussian but all exponential moments are finite. The continued fraction for the exponential function $\exp(\cdot)$ is given by*

$$e^{\lambda} \quad = \quad \frac{1}{1_{\,-}}\frac{\lambda}{1_{\,+}}\frac{\lambda}{2_{\,-}}\frac{\lambda}{3_{\,+}}\frac{\lambda}{2_{\,-}}\frac{\lambda}{5_{\,+}}\frac{\lambda}{2_{\,-}}\cdots, \ \lambda \in \mathbb{C},$$

(Cuyt et al; 2008). After truncating the above continued fraction and substituting the resulting rational function in an expression $q(q - \psi(\lambda))$, we may obtain a rational function, say $P(\lambda)/Q(\lambda)$. The characteristic functions for the extrema can be found by decomposing $P(\lambda)/Q(\lambda)$ as a product of two rational, analytic, and bounded functions in $\mathbb{C}^+$ and $\mathbb{C}^-$.

5. Conclusion and suggestion

This paper considers the problem of solving a WHF problem, approximately. As an application, it provides two approximate techniques for finding the density function of the extrema of a Lévy process. Namely, we suggest to replace *either* the jumps measure $\nu(dx)$ by a sequence of $\nu_n(dx)$, which converge (in $L^2(\mathbb{R})$, to $\nu(dx)$ *or* the characteristic exponent ψ by a sequence of approximations ψ_n, which converge to ψ, in $L^2(\mathbb{R})$. Two practical approximation procedures along several examples are given.

The methods presented in this paper can be generalized to other situations where the WHF is applicable, such as finding first/last passage time and the overshoot, the last time the extrema was archived, several kind of option pricing, etc.

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