# Optimal Stopping of Geometric Markov Renewal Processes and Pricing of European and American Options

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**Abstract:** We introduce geometric Markov renewal processes and based on them discrete-time Markov-modulated $(B, S)$-security markets and study their properties. Optimal stopping rules for these models are investigated and pricing formulas for European and American options are presented.

**Key words:** geometric Markov renewal process, optimal stopping rules, jump semi-Markov random evolutions, Markov-modulated $(B, S)$-security market, martingales, American option pricing formula, European option pricing formula

## 1 Introduction

In this paper, we study discrete-time Markov-modulated $(B, S)$-security market, that consists of a bond $B$ as a riskless asset, a stock $S$ as a risky asset and Markov renewal process that describes changes in the stock prices. The model for stock price we call geometric Markov renewal process by analogue with the geometric compound Poisson process introduced by Aase [1]. Discrete $(B, S)$-securities market with the dynamic of bond price (riskless asset) $B_n = B_{n-1}(1 + r), r > 0, B_0 > 0$, and with the dynamic of stock price (risky asset) $S_n = S_{n-1}(1 + \rho_n), S_0 > 0$, where $\rho_n$ are i.i.d. random variables such that $\rho_n = b$ with probability $p$ and $\rho_n = a$ with probabilty $q = 1 - p$, $-1 < a < r < b$, was proposed by Cox, Ross and Rubinstein in 1976 [5]. Cox, Ross, and Rubinstein [5] introduced this binomial model and derived the seminal Black-Scholes pricing formula for European call option. Our model is a generalisation of Cox, Ross and Rubinstein [5] and Aase [1] models. Also, our model for stock price as the geometric Markov renewal process is an example of discrete time jumps semi-Markov random evolution [6, 15].

We study optimal stopping problems for geometric Markov renewal processes and, based on these results, we find American option priciing formula for the discrete-time Markov-modulated $(B, S)$-security markets. Also, we present European call option pricing formula for this model. American options can be exercised at any time the holder wishes to do so before its expiry

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date. European options can be exercised at the expiry date. Since American options give the right to its holder to exercise the option at any time before the option expires, traders are more attracted by this option because of this privilege. It therefore becomes important to have knowledge on what is the best time to exercise American options in order to maximize the payoff. That is why this leads us to the study of American options as optimal stopping problems. American options pricing has been studied in many papers and books. See, for example, [3] for continuous time and [10, 11, 13, 14] for discrete-time settings, respectively. Optimal stopping problems have been studied in many papers and books as well, we mention here [4] and [12] which are the most close to our study.

Averaging, merging and double merging of geometric Markov renewal processes in series scheme and their applications in finance were studied in [17]. Diffusion approximation of geometric Markov renewal processes in series scheme with merging and double merging approaches and option pricing formulas for the limit processes were studied in [18]. Normal deviations of the geometric Markov renewal processes in series scheme for ergodic averaging and double averaging schemes are derived in [19]. We introduced Poisson averaging scheme for the geometric Markov renewal processes in [19] as well. Analogue of Black-Scholes formula for diffusion Markov-modulated $(B, S)$-security market and diffusion Markov-modulated $(B, S)$-security market with jumps have been obtained in [16].

We mention that random evolution in general and discrete-time random evolutions in particular have many applications, including finance, biology, queuing, reliability, etc. Merging of random evolutions and their applications have been studied in [6, 7, 15]. Applications of discrete-time random evolutions in reliability and DNA analysis, and their applications to difference equations and additive functionals have been considered in [2] and [8], respectively.

The paper is organized as follows. The geometric Markov renewal process (GMRP), discrete-time Markov-modulated $(B, S)$-security market and statement of the problem are presented in Section 2. In Section 3 we describe the GMRP as discrete-time jump semi-Markov random evolution and studies their martingale properties. Optimal stopping rules for GMRP are investigated in Section 4. Martingale properties of discount stock price (based on GMRP) and discount capital are considered in Section 5. American option pricing formulae for discrete time Markov-modulated $(B, S)$-security market is presented in Section 6. Here, we also consider the limit case when maturity goes to infinity (perpertual option). European option pricing formulae for discrete time Markov-modulated $(B, S)$-security market is derived in Section 7. We conclude in Section 8. Appendix contains the proofs of two Lemmas which are the complementary results for proving our main optimal stopping problem result in Section 4 (Theorem 1). The references contain only the most close to our paper articles and books.

# 2 Geometric Markov Renewal Processes and Discrete Markov-modulated $(B,S)$-Security Markets

## 2.1 Geometric Markov Renewal Process (GMRP)

Let $(x_k)_{k\in Z_+}$ be a Markov chain in phase space $(X,\mathcal{X})$ with transition probability $P(x,A)$, where $x\in X, A\in\mathcal{X}$. Let $(\tau_k)_{k\in Z_+}$ be a sequence of i.i.d. r.v. such that

$$P(\tau_{n+1}-\tau_k<t|x_n=x)=G_x(t),$$

where $x\in X, t\in R_+$. Let us set

$$\theta_{n+1}:=\tau_{n+1}-\tau_n,$$

$$\tau_n=\sum_{k=1}^{n}\theta_k,$$

and let

$$\nu(t):=\max\{n:\tau_n\le t\}$$

be a *counting process.* The following process $(x_n;\theta_n)_{n\in Z_+}$ on phase states space $X\times R_+$ is called a *Markov renewal process* (MRP) [6,7]. The process $x(t):=x_{\nu(t)}$ is called a *semi-Markov process.* (See [6, 7, 9, 15]). Let $\rho(x)$ be a bounded continuous function on $X$ such that $\rho(x)>-1$. The following functional on Markov renewal process $(x_n;\theta_n)_{n\in Z_+}$

$$S_t:=S_0\prod_{k=0}^{\nu(t)}(1+\rho(x_k)), \tag{1}$$

where $S_0$, we call *Geometric Markov Renewal Process* (GMRP). This process we call such by analogy with the geometric compound Poisson process

$$S_t=S_0\prod_{k=1}^{N(t)}(1+Y_k),$$

where $S_0>0$, $N(t)$ is a standard Poisson process, $(Y_k)_{k\in Z_+}$ are i.i.d. r.v., which is a trading model in many financial applications as a pure jump model [8].

The GMRP in (1) will be our main model in further analysis.

## 2.2 Discrete Markov-modulated $(B,S)$-Security Markets

By *discrete Markov-modulated $(B,S)$-security market* we mean the market with riskless asset (bond) $B_n$ and risky asset (stock) $S_n$ that are defined in the following way:

$$\begin{cases} B_n &= B_0(1+r)^n, \\ S_n &= S_0\prod_{k=1}^n(1+\rho(x_k)), \end{cases} \tag{2}$$

where $r > 0$ is an interest rate, $B_0 > 0, S_0 > 0$, and function $\rho(x)$ and $x_k$ are defined in section 2.1.

### 2.3 Statement of the Problem: Optimal Stopping Rule

Let $\mathcal{F}_n := \sigma\{x_k; 0 \le k \le n\}$ be a $\sigma$-algebra generated by Markov chain $x_k$. Denote by $g(s,x)$ a family of $\mathcal{R}_+ \times \mathcal{X}$-measurable functions with values in $(-\infty, +\infty]$. Let $\tau$ be a Markov time: $\tau \equiv \tau(\omega) \le N, N < +\infty$, and let define by ($\tau$ is a stopping time with respect to $\mathcal{F}_n$)

$$g(S_\tau, x_\tau) \equiv g(S_{\tau(\omega)}(\omega), x_{\tau(\omega)}(\omega))$$

the random variable which is a *gain* in the state $(S_\tau, x_\tau)$ under halting of observation at the moment $\tau$. Value $E_{s,x}g(S_\tau, x_\tau)$ is a mean gain at the initial moment $s, x$. Put

$$C_N(s,x) := \sup_\tau E_{s,x}g(S_\tau, x_\tau). \tag{3}$$

The value $C_N(s,x)$ is a price. The moment $\tau_N^*$ such that

$$E_{s,x}g(S_{\tau_N^*}, x_{\tau_N^*}) = C_N(s,x) \tag{4}$$

is called an *optimal stopping time.*

In this paper, we are going to solve the following problems:
(i) the structure of the price $C_N(s,x)$;
(ii) how to find $C_N(s,x)$;
(iii) how to find $\tau_N^*$.

## 3 GMRP as Jump Discrete-time Semi-Markov Random Evolution

Let $C_0(R_+)$ be a space of continuous function on $R_+$, vanishing on the infinity, and let us define a family of contraction operators $D(x)$ on $C_)(R_+)$ :

$$D(x)f(s) := f(s(1+\rho(x)). \tag{3}$$

Then, by definition, jump random evolution (RE) is defined as the following product:

$$V(t) = \prod_{k=1}^{\nu(t)} D(x_k).$$

That is why, from (3) we obtain:

$$V(t)f(s)=\prod_{k=1}^{\nu(t)}D(x_k)f(s)=f(s\prod_{k=1}^{\nu(t)}(1+\rho(x_k))=f(S(t)), \qquad (4)$$

where $S(t)$ is defined in (1) and $S(0)=S_0=s$. Let $Q(x,A,t)$ be a semi-Markov kernel for Markov renewal process $(x_n;\theta_n)_{n\in Z_+}: Q(x,A,t)=P(x,A)G_x(t)$. Let us define an expectation of jump RE $V(t)$ in (4):

$$u(t,x):=E_x[V(t)f(x(t))],$$

where $x(t):=x_{\nu(t)}$. Then function $u(t,x)$ satisfies the following Markov renewal equation (MRE):

$$u(t,x)-\int_0^t\int_X Q(x,dy,ds)D(y)u(t-s,y)=\bar{G}_x(t)f(x),$$

where $\bar{G}_x(t)=1-G_x(t), \qquad G_x(t):=\mathcal{P}(\theta_{n+1}\le t|X-n=x)$, $f(x)$ is a bounded and continuous function on $X$. Taking into account the above representations we obtain that

$$w(t,x,s):=E_{x,s}[f(S(t),x(t))]$$

satisfies the following MRE:

$$w(t,x,s)-\int_0^t\int_X Q(x,dy,du)w(t-u,s(1+\rho(y)),y)=\hat{G}_x(t)f(s,x),$$

where $f(s,x)$ is a bounded and continuous function on $R_+\times X$. This equation is a main tool in the investigation of limit distributions of the functional $S_T(t)=S_0\prod_{k=1}^{\nu(tT)}(1+\rho_T(x_k))$ as $T\to_{+\infty}$ . It is one of the method for obtaining all the limits for $S_T(t)$ as $T\to+\infty$. The second method is martingale method.

## 3.1 Martingale properties of GMRP

**Infinitesimal operator of the GMRP**

Let us consider the representation

$$\ln\frac{S(t)}{S_0}=\sum_{k=1}^{\nu(t)}\ln(1+\rho(x_k)).$$

To describe martingale properties of GMRP it needs to find an infinitesimal operator of the process

$$\eta(t):=\sum_{k=1}^{\nu(t)}\ln(1+\rho(x_k)).$$

Let $\gamma(t) := t - \tau_{\nu(t)}$. Let us consider the process $(x(t), \gamma(t))$ on $X \times R_+$. It is Markov process with infinitesimal operator

$$\hat{Q}f(x,t) := \frac{df}{dt} + \frac{g_x(t)}{\hat{G}_x(t)} \int_X [P(x,dy)f(y,0) - f(x,t)],$$

where $g_x(t) := \frac{dG_x(t)}{dt}$, $\hat{G}_x(t) = 1 = G_x(t)$, where $f(x,t) \in C(X \times R_+)$. Infinitesimal operator for the process $\ln S(t)$ has the form:

$$\hat{A}f(z,x) = \frac{g_x(t)}{\hat{G}_x(t)} \int_X P(x,dy)[f(z + \ln(1+\rho(y),x) - f(z,x)],$$

where $z := \ln S_0$. The process $(\ln S(t), x(t), \gamma(t))$ is a Markov process on $R_+ \times X \times R_+$ with infinitesimal operator

$$\hat{L}f(z,x,t) = \hat{A}f(z,x,t) + \hat{Q}f(z,x,t),$$

where operators $\hat{A}$ and $\hat{Q}$ are defined iabove. From here we obtain that process

$$\hat{m}(t) := f(\ln S(t), x(t), \gamma(t)) - f(z,x,0) - \int_0^t (\hat{A} + \hat{Q})f(\ln S(u), x(u), \gamma(u))du$$

is an $\hat{\mathcal{F}}_t$-martingale, where $\hat{\mathcal{F}}_t := \sigma(x(s), \gamma(s); 0 \le s \le t)$. If $x(t) := x_{\nu(t)}$ is a Markov process with kernel

$$Q(x,A,t) = P(x,A)(1 - e^{-\lambda(x)t}),$$

namely, $G_x(t) = 1 - e^{-\lambda(x)t}$, then $g_x(t) = \lambda(x)e^{-\lambda(x)t}$, $\hat{G}_x(t) = e^{-\lambda(x)t}$, and $\frac{g_x(t)}{\hat{G}_x(t)} = \lambda(x)$. That is why the operator $\hat{A}$ above has the form:

$$\hat{A}f(z) = \lambda(x) \int_X P(x,dy)[f(z + \ln(1+\rho(y))) - f(z)].$$

For pair $(\ln S(t), x(t))$ on $R_+ \times X$ we obtain that this process is Markov process with infinitesimal operator

$$\hat{L}f(z,x) = \hat{A}f(z,x) + Qf(z,x),$$

where

$$Qf(z,x) = \lambda(x) \int_X P(x,dy)(f(y) - f(x)).$$

From here it follows that process

$$m(t) := f(\ln S(t), x(t)) - f(z,x) - \int_0^t (\hat{A} + Q)f(\ln S(u), x(u))du$$

is an $\mathcal{F}_t$-martingale, where $\mathcal{F}_t := \sigma(x(u); 0 \le u \le t)$.

**Martingale property of the GMRP**

Let we have

$$S(t) = S_0 \prod_{k=1}^{\nu(t)} (1 + \rho(x_k)). \tag{5}$$

Let us define for all $t \in [0, T]$ :

$$L_t := L_0 \prod_{k=1}^{\nu(t)} h(x_k), \qquad EL_0 = 1,$$

where $h(x)$ is a bounded continuous function:

$$\int_X h(y)P(x, dy) = 1, \qquad \int_X h(y)P(x, dy)\rho(y) = 0. \tag{6}$$

If $EL_T = 1$, then process $S(t)$ in (5) is an $(\mathcal{F}_t, P^*)$-martingale, where measure $P^*$ is defined as follows

$$\frac{dP^*}{dP} = L_T,$$

and

$$\mathcal{F}_t := \sigma(x(s); 0 \leq s \leq t).$$

In discrete case we have

$$S_n = S_0 \prod_{k=1}^{\nu(t)} (1 + \rho(x_k)).$$

Let $L_n := L_0 \prod_{k=1}^{n} h(x_k)$, $\qquad EL_0 = 1$, where $h(x)$ is defined in (6). If $EL_N = 1$, then $S_n$ is an $(\mathcal{F}_t, P^*)$-martingale, where $\frac{dP^*}{dP} = L_N$, and $\mathcal{F}_n := \sigma(x_k; 0 \leq k \leq n)$.

# 4 Optimal Stopping Rules for GMRP

Let

$$Qg(s, x) := \max\{g(s, x); Tg(s, x)\}, \tag{7}$$

where operator $T$ is defined by

$$Tg(s, x) := E_{s,x} g(S_1, x_1) = E_{s,x} g(s(1 + \rho(x_1)), x_1), \tag{8}$$

$S_0 = s$. Let also

$$C_n(s, x) := \sup_{\tau \in \mathcal{M}_g(n)} E_{s,x} g(S_\tau, x_\tau), \tag{9}$$

where

$$\mathcal{M}_g(n) := \{\tau : \tau(\omega) \leq n; n < +\infty; E_{s,x} g_-(S_\tau, x_\tau) < +\infty, x \in X\}, \tag{10}$$

and

$$g_-(s,x) := -\min(g(s,x),0).$$

**Theorem 1.** Let $g(s,x)$ be such that $E_{s,x}g_-(S_1,x_1) < +\infty$. Then:
1)

$$C_n(s,x) = Q_n g(s,x), n = 0,1,2,...; \tag{11}$$

2)

$$C_n(s,x) = \max\{g(s,x); TC_{n-1}(s,x)\}, \tag{12}$$

where $C_0(s,x) = g(s,x)$;

3) the moment (stopping time)

$$\tau_n^* := \min\{0 \le m \le n : C_{n-m}(S_m,x_m) = g(S_m,x_m)\} \tag{13}$$

is an optimal stopping time and

$$C_n(s,x) = E_{s,x}g(S_{\tau_n^*},x_{\tau_n^*}). \tag{14}$$

**Remark 1.** From the definition of $Q$ in (7) it follows that

$$Q_n g(s,x) = \max\{Q^{n-1}g(s,x), TQ^{n-1}g(s,x)\}, n = 1,2,...,$$

where $Q^0 g(s,x) = g(s,x)$. Also,

$$Q_n g(s,x) = \max\{g(s,x); TQ^{n-1}g(s,x)\}. \tag{13}$$

Really, for $n = 2$ we obtain

$$Q_2 g(s,x) = \max\{Qg(s,x); TQg(s,x)\} =$$

$$\max\{\max\{g(s,x); Tg(s,x)\}; T\{\max\{g(s,x); Tg(s,x)\}\} =$$

$$\max\{g(s,x); T\{\max\{g(s,x); Tg(s,x)\}\} =$$

$$\max\{g(s,x); TQg(s,x)\}.$$

If $n = 2$, the formula (13) is true. To prove (13) for all $n$ we use the method of mathematical induction.

The proof of **Theorem 1** is based on the two following lemmas.

**Lemma 1.** For every $\tau \in \mathcal{M}_g(n)$ :

$$E_{s,x}g(S_\tau,x_\tau) \le Q_n g(s,x), \forall x \in X, \tag{14}$$

and, hence,

$$C_n(s,x) \le Q_n g(s,x). \tag{15}$$

**Lemma 2.** For all $n = 0,1,2,...$

$$Q_n g(s,x) = E_{s,x}g(S_{\sigma_n},x_{\sigma_n}), \tag{16}$$

where

$$\sigma_n := \min\{0 \le k \le n : Q_{n-k}g(S_k,x_k) = g(S_k,x_k)\}. \tag{17}$$

The proofs of these lemmas will be done in the **Appendix**.

**Proof of the Theorem 1.** From (15) and (16) it follows that:

$$C_n(s,x) \leq Q^n g(s,x) = E_{s,x} g(S_{\sigma_n}, x_{\sigma_n}).$$

Obviously,

$$C_n(s,x) \geq E_{s,x} g(S_{\sigma_n}, x_{\sigma_n}),$$

as $C_n(s,x)$ is a price, namely, $\sup E_{s,x} g$. In this way, for all $n = 0, 1, 2, ...$

$$C_n(s,x) = Q^n g(s,x) = E_{s,x} g(S_{\sigma_n}, x_{\sigma_n}),$$

and, hence, the stopping time $\sigma_n$ (=$\tau_n^*$) is an optimal one, where $\sigma_n$ is defined in (17). Statement 2) (see (12)) follows from 1), (6) and (13). The **Theorem 1** is proved.

**Remark 2.** Put

$$\mathcal{D}_n^N := \{(s,x) : C_{N-n}(s,x) = g(s,x)\}, 0 \leq n \leq N. \tag{18}$$

Stopping time $\tau_N^*$ (optimal) in (4) is described in terms of "stopping sets" $\mathcal{D}_n^N$ in (18):

$$\tau_N^* = \min\{0 \leq n \leq N : (S_k, x_k) \in \mathcal{D}_n^N\}, \tag{19}$$

that follows from **Theorem 1.** Namely, if $(s,x) \in \mathcal{D}_0^N$ $((s,x) := (S_0, x_0))$, then optimal rule orders instant stopping. If $(s,x) \notin \mathcal{D}_0^N$, then we continue our observation, and, depending on $(S_1, x_1)$, either stop our observation (if $(S_1, x_1) \in \mathcal{D}_1^N$) or continue our next observation (if $(S_1, x_1) \notin \mathcal{D}_1^N$), and so on. Obviously, that observation process will stop at the moment $N$ as $\mathcal{D}_N^N = R_+ \times X$.

Let us consider the "set of continuation of observations":

$$F_n^N = R_+ \times X \setminus \mathcal{D}_n^N; o \leq n \leq N.$$

Obviously: $F_N^N = \emptyset$ and

$$F_{N-1}^N = \{(s,x) : C_1(s,x) > g(s,x)\} =$$

$$\{(s,x) : Qg(s,x) > g(s,x)\} = \{(s,x) : Tg(s,x) > g(s,x)\} =$$

$$\{(s,x) : (T-I)g(s,x) > 0\},$$

where $I$ is an identity operator. As $g(s,x) \leq C_1(s,x) \leq ... \leq C_N(s,x)$, then $F_n^N$ satisfies the following relation:

$$\emptyset = F_N^N \subseteq F_{N-1}^N \subseteq ... \subseteq F_0^N.$$

In particular,

$$F_0^N = \{(s,x) : C_N(s,x) > g(s,x) \supseteq \{(s,x) : (T-I)g(s,x) > 0\}.$$

**Remark 3.** To find the stopping time $\tau_N^*$ it is necessary to know the prices $C_n(s,x)$, for all $n : 0 \leq n \leq N$. In this way, to solve a problem on optimal stopping in the following class $\mathcal{M}(N) := \{\tau : \tau \leq N; N < +\infty\}$ it is necessary to solve the stopping problems successively in the following classes $\mathcal{M}(1), ..., \mathcal{M}(N-1)$. Respectively, the prices are founded with the help of iterations of operator $Q : C_1(s,x) = Qg(s,x); ..., C_{N-1}(s,x) = Q^{N-1}g(s,x)$, or

$$C_n(s,x) = \max\{g(s,x); TC_{n-1}(s,x)\}.$$

# 5 Martingale Properties of Discount Price $S_n/B_n$ and Discount Capital $X_n/B_n$

Let us define a discounted stock price:

$$D_n := S_n/B_n, \tag{20}$$

where $S_n$ and $B_n$ are defined in (2). Then

$$E_x[D_n/\mathcal{F}_n] = E_x[D_{n-1}\frac{1+\rho(x_n)}{1+r}/\mathcal{F}_n] =$$

$$D_{n-1}E_{x_{n-1}}[\frac{1+\rho(X_n)}{1+r}] = D_{n-1},$$

if

$$E_{x_{n-1}}(\frac{1+\rho(x_n)}{1+r}) = 1,$$

or

$$\int_X P(x,dy)\rho(y) = r, \tag{21}$$

where $P(x,dy)$ are transition probabilities of Markov chain $(x_n)_{n\in Z_+}$. In this way, the sequence $D_n$ is an $\mathcal{F}_n$-martingale if (21) is fulfilled. Let

$$X_n := \beta_n B_n + \gamma_n S_n,$$

with $(\beta_n, \gamma_n)$ being an investor's portfolio (which is $\mathcal{F}_{n-1}$-measurable), and $(B_n, S_n)$ being defined in (2).

We suppose that our portfolio $(\beta_n, \gamma_n)$ is self-financing, e.g., $\pi_n := (\beta_n, \gamma_n) \in SF$ :

$$\triangle\beta_n B_n + \triangle\gamma_n S_n = 0. \tag{22}$$

Then from (22) it follows that

$$\triangle X_n = \beta_n \triangle B_n + \gamma_n \triangle S_n = \beta_n(rB_{n-1}) + \gamma_n(\rho(x_n)S_n) \tag{23}$$

Put

$$M_n := \frac{X_n}{B_n} \tag{24}$$

for discounted capital. Then from (23) it follows that

$$\triangle M_n = \frac{X_n}{B_n} - \frac{X_{n-1}}{B_{n-1}} =$$
$$\frac{X_n - X_{n-1} - rX_{n-1}}{B_n} =$$
$$\frac{\gamma_n S_{n-1}(\rho(x_n) - r)}{B_n}. \tag{25}$$

Let

$$m_n := \sum_{k=1}^{n} (\rho(x_k) - r)$$

and

$$\triangle m_n = (\rho(x_n) - r). \tag{26}$$

From (29) it follows that for $n \geq 1$

$$M_n = M_0 + \sum_{k=1}^{n} \frac{\gamma_n S_{k-1}}{B_k} \triangle m_k. \tag{27}$$

Let $E^*$ be an expectation by $P(x, A)$ as in (21). Then $m_n$ in (26) is an $\mathcal{F}_n$-martingale by $E^*$. Values $\gamma_k$ and $S_{k-1}$ are $\mathcal{F}_{k-1}$-measurable, hence, sequence $(M_n, \mathcal{F}_n)$ is also martingale, where $M_n$ is defined in (24) (see also (27)). That is why

$$E^* M_N = M_0. \tag{28}$$

From (28) and (24) we obtain:

$$E^*(1+r)^{-N} X_N = x, X_0 = x. \tag{29}$$

A portfolio $\pi_n := (\beta_n, \gamma_n)$ is a $(x, f_N)$-hedge, if for given $x > 0$ and nonnegative function $f_N(x) := f_N(S_0, S_1, ..., S_N, x_1, ..., x_N)$, $X_0 = x$ and

$$X_N \geq f_N(x). \tag{30}$$

If

$$X_N = f_N(x)$$

then $\pi_n$ is a minimal $(x, f_N)$-hedge. Let $\Pi(x, f_N)$ be a family of all $(x, F_N)$-hedges $\pi \in SF$. From (29) and (30) we obtain that if $\pi_n \in \Pi(x, f_N)$, then

$$x \leq E^*(1+r)^{-N} f_N(x). \tag{31}$$

If the hedge $\pi_n$ is a minimal one, then

$$x = E^*(1+r)^{-N} f_N(x),$$

that follows from (31) and (29). In this way we obtain the following result.

**Lemma 3.** Let on discrete $(B, S, X)$-securities market the portfolio $\pi_n = (\beta_n, \gamma_n) \in SF$ is $(x, f_N)$-hedge. Then $x \geq E^*(1+r)^{-N} f_N(x)$. If $(x, f_N)$-hedge $\pi_n$ is a minimal one, then $x = E^*(1+r)^{-N} f_N(x)$, where $E^*$ is an expectation by $P(x, A)$ in (21), $x \in X, A \in \mathcal{X}$.

# 6 American Option Pricing Formulae for Discrete Markov-modulated $(B, S)$-Security markets

As $M_n = \frac{X_n}{B_n}$ is an $\mathcal{F}_n$-martingale, and, $N < +\infty$, then for any stopping time $\tau, \tau \leq N$, we have $E^* M_\tau = M_0$, namely,

$$E^*(1+r)^{-\tau} X_\tau = X_0. \tag{32}$$

Let us suppose that portfolio $\pi$ is $(x, f, N)$-hedge, namely, $X_0 = x$ and $X_n \geq f_n(S_0, S_1, ..., S_n, x_1, ..., x_n)$ for any $0 \leq n \leq N$. Then from (38) we obtain that

$$x \geq \sup_{0 \leq \tau \leq N} E^*(1+r)^{-\tau} f_\tau(x).$$

If the $(x, f, N)$-hedge $\pi_n$ is a minimal one (e.g., there exists a stopping time $\sigma$ such that for all $\omega \in \Omega$ $X_\sigma = f_\sigma(x)$,) then $x = X_0 = E^*(1+r)^{-\sigma} X_\sigma = E^*(1+r)^{-\sigma} f_\sigma$, and, hence

$$x = \sup_{0 \leq \tau \leq N} E^*(1+r)^{-\tau} f_\tau(x). \tag{33}$$

From here we obtain the following result: the *rational price $C_N^*$ of American option with maturity date $N$ and system of nonnegative cost functions $f(x) = (f_n(x))_{0 \leq n \leq N}$ for discrete $(B, S, X)$-securities market is defined by the following formula:*

$$C_N^* = \sup_{0 \leq \tau \leq N} E^*(1+r)^{-\tau} f_\tau(x). \tag{34}$$

A Stopping time $\tau^*$ is rational if and only if

$$E^*(1+r)^{-\tau^*} f_{\tau^*}(x) = \sup_{0 \leq \tau \leq N} E^*(1+r)^{-\tau} f_\tau(x).$$

In this way, the problem of finding of a rational price $C_N^*$ and rational stopping time is solved by the solving of one problem: *optimal stopping* of

$$\sup_\tau E^*(1+r)^{-\tau} f_\tau(x).$$

Let

$$f_n := \beta^n g(S_n, x_n), \tag{35}$$

where $0 < \beta \leq 1$. We will solve this problem in the last case of (35). Let us put

$$V_n := \sup_{0 \leq \tau \leq N} E^*_{s,x}(1+r)^{-\tau} \beta^\tau g(S_\tau, x_\tau), \tag{45}$$

and

$$Tg(s, x) := E^*_{s,x} g(S_1, x_1).$$

Then

$$Tg(s,x) = E^*_{s,x} g(s(1+\rho(x_1)), x_1) = \int_X P(x,dy) g(s(1+\rho(y)), y). \tag{37}$$

Put

$$Q_\beta g(s,x) := \max\{g(s,x), (1+r)^{-1}\beta T g(s,x)\}.$$

Then, with respect to the previous results (see Section 2, Theorem 1) we obtain:

1)the function $V_n(s,x)$ in (36) is represented in explicit form:

$$V_n(s,x) = Q^n_\beta g(s,x),$$

where $Q^n_\beta$ is an $n-th$ power of operator $Q_\beta$;

2)

$$V_n(s,x) = \max\{g(s,x); (1+r)^{-1}\beta T V_{n-1}(s,x)\};$$

$$V_0(s,x) = g(s,x);$$

3) the stopping time

$$\tau_n := \min\{0 \le m \le n : V_{n-m}(S_m, x_m) = g(S_m, x_m)\} \tag{38}$$

is an optimal one:

$$E^*_{s,x}((1+r)^{-1}\beta)^{\tau_n} g(S_{\tau_n}, x_{\tau_n}) = V_n(s,x). \tag{39}$$

Let

$$\mathcal{D}_m := \{(s,x) : V_m(s,x) = g(s,x)\}, \tag{40}$$

$$F_m := \{(s,x) : V_m(s,x) > g(s,x)\} = R_+ \times X \setminus \mathcal{D}_m. \tag{41}$$

We note that

$$F_n \supseteq F_{n-1} \supseteq ... \supseteq F_0 = \emptyset;$$

$$\mathcal{D}_n \subseteq \mathcal{D}_{n-1} \subseteq ... \subseteq \mathcal{D}_0 = R_+ \times X.$$

From (38) and (39) it follows that stopping time $\tau_m = \min\{0 \le m \le n : (S_m, x_m) \in \mathcal{D}_{n-m}\}$ is an optimal one. Sets $\mathcal{D}_n, \mathcal{D}_{n-1}, ..., \mathcal{D}_0$ are "stopping sets", and sets $F_n, F_{n-1}, ..., F_0$ are "sets of continuation of observations".

From (40)-(41) it obviously follows that

$$\mathcal{D}_0 = R_+ \times X,$$

$$\mathcal{D}_1 = \{(s,x) : Q_\beta g(s,x) = g(s,x)\},$$

$$\mathcal{D}_2 = \{(s,x) : Q^2_\beta g(s,x) = g(s,x)\}$$

and so on.

**Example.** $f_n(s,x) = \beta^n g(s,x)$, where $g(s,x) = (s-1)^+ h(x)$, $h(x)$ is a continuous bounded function on $X$.

### 6.1 Limit Case: $N \to +\infty$.

As $V_0(s,x) \leq V_1(s,x) \leq ... \leq ...$, then there exists a limit $\lim_{n\to+\infty} V_n(s,x) = V(s,x)$. This function has a properties:

1)

$$V(s,x) = \sup_\tau E^*_{s,x}((1+r)^{-1}\beta)^\tau g(S_\tau, x_\tau); \tag{42}$$

2)

$$V(s,x) = \max\{g(s,x), (1+r)^{-1}\beta T V(s,x)\};$$

3)$V(s,x)$ is the least of functions $u(s,x) \geq 0$ such that: $u(s,x) \geq g(s,x)$ and $u(s,x) \geq ((1+r)^{-1}\beta T u(s,x;$

4)the stopping time

$$\tau_\infty := \inf\{n : V(S_n, x_n) = g(S_n, x_n)\}$$

is an optimal one, e.g.,

$$V(s,x) = E^*_{s,x}((1+r)^{-1}\beta)^{\tau_\infty} g(S_\infty, x_\infty).$$

If $C^*(s,x)$ is a set of continuation of observations, then $V(s,x) > g(s,x)$ and from (42) it follows that: $V(s,x) = (1+r)^{-1}\beta T V(s,x)$, namely,

$$V(s,x) = (1+r)^{-1}\beta \int_X P(x,dy) V(s(1+\rho(y)), y).$$

# 7 European Option Pricing Formula for Discrete Markov-Modulated $(B,S)$-Security Markets

Let us consider European call option, where the dynamic of stock price is discribed by discrete time Markov-modulated $(B,S)$-security market:

$$\begin{cases} B_n &= B_0(1+r)^n, \\ S_N &= S_0 \prod_{k=1}^N (1+\rho(x_k)), \end{cases}$$

where $N$ is an maturity date. Let

$$f(S_N) = (S_N - K)^+,$$

where $K$ is a strike price. From Section 5 it follows that optimal (or rational) price of European call option is equal to:

$$C_N(x) = E^*(1+r)^{-N} f(S_N) = E^*(1+r)^{-N}(S_N - K)^+, \tag{43}$$

where $E^*$ is an expectation by $P(x,dy)$ such that $\int_X P(x,dy)\rho(y) = r$. From (43) we obtain:

$$\begin{aligned} C_N(x) &= E^*(1+r)^{-N}(S_N - K)^+ \\ &= (1+r)^{-N} \int_X ... \int_X (S_0 \prod_{i=1}^N (1+\rho(x_i)) - K)^+ P(y_{i-1}, dy_i). \end{aligned} \tag{44}$$

Let us consider the geometric Markov renewal process:

$$S(t) = S_0 \prod_{k=1}^{\nu(t)} (1 + \rho(x_k)),$$

where $(x_k; \theta_k)_{k \in Z_+}$ is a Markov renewal process, $x_k \in X$, $\theta_k \in R_+$, $\nu(t) := \max\{n : \tau_n \leq t\}$ is a counting process, $\tau_n := \sum_{k=1}^{n} \theta_k$, $\theta_0 = 0$. Let

$$f(S(T)) = (S(T) - K)^+,$$

where $S(T) = S_0 \prod_{k=1}^{\nu(T)} (1 + \rho(x_k))$.

The price of an European call option is:

$$C_T(x) = E^*[f(S(T))(1+r)^{-N}] = E^*[(1+r)^{-N} E^*[f(S(T))|\nu(T)]] =$$

$$\sum_{k=0}^{\infty} \mathcal{P}\{\nu(T) = k\} \int_X ... \int_X (S_0 \prod_{i=1}^{k} (1 + \rho(y_i)) - K)^+ P(y_{i-1}, dy_i). \qquad (45)$$

In the case of Poisson process $\nu(t) \equiv N(t)$ with intensity $\lambda > 0$ of jumps we obtain from (44) and (45):

$$C_T(x) = \sum_{k=0}^{\infty} \frac{e^{-\lambda T} (\lambda T)^k}{k!} \int_X ... \int_X [S_0 \prod_{i=1}^{k} (1 + \rho(y_i)) - K]^+ P(y_{i-1}, dy_i). \quad (46)$$

# 8 Conclusion and Future Work

We introduced geometric Markov renewal processes and based on them discrete-time Markov-modulated $(B, S)$-security markets and studied their properties. Optimal stopping rules for these models have been investigated and pricing formulas for European and American options have been presented. Our future work will be associated with optimal stopping rules for discrete time jump semi-Markov random evolutions and implementations of obtained results for discrete time Markov-modulated $(B, S)$-security markets.

# Appendix

**Proof of Lemma 1.** If $n = 0$, then (14) is obviously follows. Let now $\tau \in \mathcal{M}_g(n), n > 0$. Put $B := \{\omega : \tau(\omega) = n\}$. Then $B = \Omega \setminus \sum_{i=0}^{n-1} \{\tau = i\} \in \mathcal{F}_{n-1}$, and

$$E_{s,x} g(S_\tau, x_\tau) = E_{s,x} \mathbf{1}_{\bar{B}} g(S_\tau, x_\tau + \sum_{s,x} \mathbf{1}_B g(S_\tau, x_\tau =$$

$$E_{s,x} \mathbf{1}_{\bar{B}} g(S_{\tau \wedge (n-1)}, x_{\tau \wedge (n-1)}) + E_{s,x} \mathbf{1}_B g(S_n, x_n) =$$

$$E_{s,x} \mathbf{1}_{\bar{B}} g(S_{\tau \wedge (n-1)}, x_{\tau \wedge (n-1)}) + E_{s,x} \{\mathbf{1}_B E_{s,x} \{g(S_n, x_n)|\mathcal{F}_{n-1}\}\} =$$

$$E_{s,x}\mathbf{1}_{\bar{B}}g(S_{\tau\wedge(n-1)},x_{\tau\wedge(n-1)})+E_{s,x}\mathbf{1}_{B}E_{S_{n-1},x_{n-1}}g(S_1,x_1)=$$
$$E_{s,x}\mathbf{1}_{\bar{B}}g(S_{\tau\wedge(n-1)},x_{\tau\wedge(n-1)})+E_{s,x}\mathbf{1}_{B}E_{S_{\tau\wedge(n-1)},x_{\tau\wedge(n-1)}}g(S_1,x_1)\leq$$
$$E_{s,x}\max[g(S_{\tau\wedge(n-1)},x_{\tau\wedge(n-1)}),E_{S_{\tau\wedge(n-1)},x_{\tau\wedge(n-1)}}g(S_1,x_1)]=$$
$$E_{s,x}Qg(S_{\tau\wedge(n-1)},x_{\tau\wedge(n-1)}).$$

So

$$E_{s,x}g(S_\tau,x_\tau)\leq E_{s,x}Qg(S_{\tau\wedge(n-1)},x_{\tau\wedge(n-1)}). \tag{47}$$

Finally, we obtain from (47):

$$E_{s,x}g(S_\tau,x_\tau)\leq E_{s,x}Qg(S_{\tau\wedge(n-1)},x_{\tau\wedge(n-1)})\leq$$
$$E_{s,x}Q^2g(S_{\tau\wedge(n-2)},x_{\tau\wedge(n-2)})\leq ...$$
$$E_{s,x}Q^ng(S_{\tau\wedge 0},x_{\tau\wedge 0})=Q^ng(s,x),$$

that proves (14) and (15), and **Lemma 1** is proved.

**Proof of Lemma 2.** The Proof will be done by the method of mathematical induction. If $n=0$, then statement of **Lemma 2** is obviously. Let the equality (16) is true for some $n\geq 0$. Let us show that (16) is also true for $(n+1)$. Let us fix the point $(s,x)\in R_+\times X$. Then, if $\mathcal{P}\{\sigma_{n+1}=0\}=1$, then by (17) $\mathcal{P}\{Q^{n+1}g(S_0,x_0)=g(S_0,x_0)\}=1$, and, hence, $Q^{n+1}g(s,x)=g(s,x)=E_{s,x}g(S_{\sigma_{n+1}},x_{\sigma_{n+1}})$.

Let now $\mathcal{P}_{s,x}\{\sigma_{n+1}=0\}<1$. Then, since $\{\sigma_{n+1}=0\}\in\mathcal{F}_0$, by law of "0" and "1" $\mathcal{P}_{s,x}\{\sigma_{n+1}=0\}=0$, and, hence $\mathcal{P}_{s,x}\{\sigma_{n+1}\geq 1\}=1$. Let show that in this case $\sigma_{n+1}=1+\theta_1\sigma_n$, where $\theta_1\sigma_n:=\sigma_n(\theta_1\omega):=\sigma_{n+1}(\omega)$. Really,

$$\theta_1\sigma_n(\omega)=\theta_1\min\{0\leq k\leq n:Q^{n-k}g(S_k(\omega),x_k(\omega))=g(S_k(\omega),x_k(\omega))\}=$$
$$\min\{0\leq k\leq n:Q^{n-k}g(S_k(\theta_1\omega),x_k(\theta_1\omega))=g(S_k(\theta_1\omega),x_k(\theta_1\omega))\}=$$
$$\min\{0\leq k\leq n:Q^{n-k}g(S_{k+1}(\omega),x_{k+1}(\omega))=g(S_{k+1}(\omega),x_{k+1}(\omega))\}=$$
$$\min\{0\leq k\leq n:Q^{n+1-(k+1)}g(S_{k+1}(\omega),x_{k+1}(\omega))=g(S_{k+1}(\omega),x_{k+1}(\omega))\}.$$

Hence,

$$1+\theta_1\sigma_n(\omega)=\min\{1\leq k+1\leq n+1:Q^{n+1-(k+1)}g(S_{k+1},x_{k+1})=g(S_{k+1},x_{k+1})\}=$$
$$\min\{1\leq l\leq n+1:Q^{n+1-l}g(S_l,x_l)=g(S_l,x_l)\}=\sigma_{n+1}(\omega),$$

where the last equality follows from definition of $\sigma_{n+1}$ in (17) and preposition $\mathcal{P}\{\sigma_{n+1}\geq 1\}=1$. Here, $Q^{n+1}g(s,x)>g(s,x)$. From here, and induction preposition and from (13) we obtain:

$$Q^{n+1}g(s,x)=\max\{g(s,x),E_{s,x}Q^ng(S_1,x_1)\}=E_{s,x}E_{s,x}g(S_{\sigma_n},x_{\sigma_n})=$$
$$E_{s,x}\theta_1g(S_{\sigma_n},x_{\sigma_n})=$$
$$E_{s,x}g(S_{1+\theta_1\sigma_1},x_{1+\theta_1\sigma_n})=E_{s,x}g(S_{\sigma_{n+1}},x_{\sigma_{n+1}}),$$

that complete the proof of **Lemma 2.**

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